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# FIXED POINT THEOREMS FOR NONLINEAR MAPPINGS RELATED TO MAXIMAL MONOTONE OPERATORS IN BANACH SPACES

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**ABSTRACT.** In this paper, we study the existence of fixed points of nonspreading mappings and the approximation of fixed points of firmly nonexpansive type mappings in Banach spaces. Applications to a proximal point algorithm for monotone operators in Banach spaces are also included.

## 1. INTRODUCTION

Let  $E$  be a (real) Banach space and let  $T$  be a mapping from  $C$  into itself. We denote the set of fixed points of  $T$  by  $F(T)$ , that is,  $F(T) = \{z \in C : Tz = z\}$ . The mapping  $T$  is said to be *nonexpansive* if

$$(1.1) \quad \|Tx - Ty\| \leq \|x - y\|$$

for all  $x, y \in C$ . It is also said to be *firmly nonexpansive* [4] if

$$(1.2) \quad \|Tx - Ty\| \leq \|r(x - y) + (1 - r)(Tx - Ty)\|$$

for all  $x, y \in C$  and  $r > 0$ ; see also [5, 11, 19].

The fixed point problem for nonexpansive mappings in Hilbert spaces is related to the problem of finding zero points of maximal monotone operators in the space. In fact, if  $H$  is a Hilbert space and  $A \subset H \times H$  is a maximal monotone operator, then for each  $r > 0$ , the resolvent  $J_r$  of  $A$  defined by

$$(1.3) \quad J_r x = \{z \in H : x \in z + rAz\}$$

for all  $x \in H$  is a single-valued firmly nonexpansive mapping from  $H$  into itself and the equality  $F(J_r) = A^{-1}0$  holds; see [31, 32].

There are two generalizations of the class of maximal monotone operators in Hilbert spaces to Banach spaces. One of them is the class of  $m$ -accretive operators and the other is that of maximal monotone operators. It is known that the class of resolvents of accretive operators in Banach spaces coincides with that of firmly nonexpansive mappings. See [5, 24] on convergence theorems and [12, 29] on fixed point theorems for firmly nonexpansive mappings in Banach spaces.

Let  $E$  be a smooth Banach space and let  $J$  be the (normalized) duality mapping from  $E$  into  $E^*$ . Following [1, 15], let  $\phi$  be the mapping from  $E \times E$  into  $[0, \infty)$  defined by

$$(1.4) \quad \phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

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for all  $x, y \in E$ . It is easy to see that  $\phi(x, y) \geq (\|x\| - \|y\|)^2 \geq 0$  for all  $x, y \in E$ . Let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a mapping from  $C$  into itself. Then we say that  $T$  is *nonspreading* [16] if

$$(1.5) \quad \phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all  $x, y \in C$ . We also say that  $T$  is *firmly nonexpansive type* [17] if

$$(1.6) \quad \langle Tx - Ty, JTx - JTy \rangle \leq \langle Tx - Ty, Jx - Jy \rangle$$

for all  $x, y \in C$ . It is easy to verify that if  $E$  is a smooth, strictly convex and reflexive Banach space and  $A \subset E \times E^*$  is a maximal monotone operator, then for each  $r > 0$ , the resolvent  $Q_r$  of  $A$  defined by

$$(1.7) \quad Q_r x = \{z \in E : Jz \in Jx + rA\} \quad (= (J + rA)^{-1}Jx)$$

for all  $x \in E$  is a firmly nonexpansive type mapping. In fact, if  $x, y \in E$  and  $r > 0$ , then it follows from

$$(1.8) \quad \left( Q_r x, \frac{Jx - JQ_r x}{r} \right), \left( Q_r y, \frac{Jy - JQ_r y}{r} \right) \in A$$

and the monotonicity of  $A$  that

$$(1.9) \quad \left\langle Q_r x - Q_r y, \frac{Jx - JQ_r x}{r} - \frac{Jy - JQ_r y}{r} \right\rangle \geq 0.$$

This gives us that  $Q_r$  is a firmly nonexpansive type mapping.

The purpose of the present paper is to state some results for nonspreading or firmly nonexpansive type mappings in Banach spaces which were recently obtained in [16, 17]. Our paper is organized as follows: In Section 2, we state some definitions and results needed in this paper. After that, we show that every firmly nonexpansive type mapping is nonspreading. In Section 3, we obtain fixed point theorems for nonspreading mappings in Banach spaces. In Section 4, we first show that every nonspreading mapping (resp. firmly nonexpansive type mapping) with a fixed point is relatively nonexpansive (resp. strongly relatively nonexpansive). Then we show a weak convergence theorem for a single firmly nonexpansive type mapping in Banach spaces. In Section 5, we apply our results to a proximal point algorithm in Banach spaces.

## 2. PRELIMINARIES

Throughout the present paper, every linear space is real. The sets of positive integers and real numbers are denoted by  $\mathbb{N}$  and  $\mathbb{R}$ , respectively. Let  $E$  be a Banach space with norm  $\|\cdot\|$  and let  $E^*$  be the dual space of  $E$ . Then the value of  $x^* \in E^*$  at  $x \in E$  is denoted by  $\langle x, x^* \rangle$ . The strong and weak convergence of a sequence  $\{x_n\}$  of  $E$  to  $x \in E$  are denoted by  $x_n \rightarrow x$  and  $x_n \rightharpoonup x$ , respectively. The *duality mapping*  $J$  from  $E$  into  $2^{E^*}$  is defined by  $Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$  for all  $x \in E$ . The space  $E$  is said to be *smooth* if the limit

$$(2.1) \quad \lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all  $x, y \in S(E)$ , where  $S(E)$  is the unit sphere of  $E$ . In this case, the norm of  $E$  is said to be *Gâteaux differentiable*. The norm of  $E$  is also said to be *uniformly Gâteaux differentiable* if for all  $y \in S(E)$ , the limit (2.1) converges uniformly in  $x \in S(E)$ . The space  $E$  is said to be *strictly convex* if  $\|(x + y)/2\| < 1$  whenever  $x, y \in S(E)$  and  $x \neq y$ .

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It is also said to be *uniformly convex* if for all  $\varepsilon \in (0, 2]$ , there exists  $\delta > 0$  such that  $\|(x+y)/2\| \leq 1 - \delta$  whenever  $x, y \in S(E)$  and  $\|x - y\| \geq \varepsilon$ . The duality mapping  $J$  from a smooth Banach space  $E$  into  $E^*$  is said to be *weakly sequentially continuous* if  $\{Jx_n\}$  converges to  $Jx$  in the weak\* topology of  $E^*$  whenever  $\{x_n\}$  is a sequence of  $E$  such that  $x_n \rightarrow x$ . We know the following; see, for instance, [10, 32]:

- (1) If  $E$  is smooth, then  $J$  is single-valued;
- (2) if  $E$  is reflexive, then  $J$  is onto;
- (3) if  $E$  is strictly convex, then  $J$  is one-to-one.

Let  $E$  be a Banach space and let  $A$  be a subset of  $E \times E^*$ . We always identify the set  $A$  with the mapping  $\hat{A} : E \rightarrow 2^{E^*}$  defined by  $\hat{A}x = \{x^* \in E^* : (x, x^*) \in A\}$  for all  $x \in E$ . Then the domain and the range of  $A$  are defined by  $D(A) = \{x \in E : Ax \neq \emptyset\}$  and  $R(A) = \bigcup_{x \in D(A)} Ax$ , respectively. The operator  $A$  is said to be *monotone* if  $\langle x - y, x^* - y^* \rangle \geq 0$  whenever  $(x, x^*), (y, y^*) \in A$ . A monotone operator  $A$  is also said to be *maximal monotone* if there is no other monotone operator  $B \subset E \times E^*$  such that  $A \subset B$  and  $A \neq B$ .

Let  $E$  be a smooth Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Then an element  $u$  of  $C$  is said to be an *asymptotic fixed point* [23] of  $T$  if there exists a sequence  $\{x_n\}$  of  $C$  such that  $x_n \rightarrow u$  and  $\|x_n - Tx_n\| \rightarrow 0$ . The set of asymptotic fixed points of  $T$  is denoted by  $\hat{F}(T)$ . The mapping  $T$  is said to be *relatively nonexpansive* [20, 21] if the following conditions are satisfied:

- (1)  $F(T)$  is nonempty;
- (2)  $\hat{F}(T) = F(T)$ ;
- (3)  $\phi(u, Tx) \leq \phi(u, x)$  for all  $(u, x) \in F(T) \times C$ ;

see also [6, 7, 8, 9, 23] for similar classes of nonlinear operators. A relatively nonexpansive mapping  $T$  from  $C$  into itself is also said to be *strongly relatively nonexpansive* [23] if  $\phi(Tz_n, z_n) \rightarrow 0$  whenever  $\{z_n\}$  is a bounded sequence of  $C$  such that  $\phi(p, z_n) - \phi(p, Tz_n) \rightarrow 0$  for some  $p \in F(T)$ .

Let  $E$  be a smooth, strictly convex and reflexive Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Then for all  $x \in E$ , there exists a unique  $x_0 \in C$  (denoted by  $\Pi_C x$ ) such that  $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$ . The mapping  $\Pi_C$  is said to be the *generalized projection* from  $E$  onto  $C$ ; see [1, 15].

We know the following lemma:

**Lemma 2.1** ([17]). *Let  $E$  be a smooth Banach space, let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a mapping from  $C$  into itself. Then the following are equivalent:*

- (1) *The mapping  $T$  is firmly nonexpansive type;*
- (2)  *$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x)$  for all  $x, y \in C$ .*

By Lemma 2.1, we know that every firmly nonexpansive type mapping is nonspreading:

**Corollary 2.2** ([16]). *Let  $E$  be a smooth Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Then every firmly nonexpansive type mapping from  $C$  into itself is nonspreading.*

We also know the following lemma, which shows that the class of firmly nonexpansive type mappings coincides with that of resolvents of monotone operators in Banach spaces:

**Lemma 2.3** ([16]). *Let  $E$  be a smooth, strictly convex and reflexive Banach space, let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a mapping from  $C$  into itself. Then the following are equivalent:*

- (1) *The mapping  $T$  is firmly nonexpansive type;*
- (2) *there exists a monotone operator  $A \subset E \times E^*$  such that  $D(A) \subset C \subset J^{-1}R(J + A)$  and  $Tx = (J + A)^{-1}Jx$  for all  $x \in C$ .*

As direct consequences of Lemmas 2.1 and 2.3, we obtain the following corollaries:

**Corollary 2.4** ([17]). *Let  $E$  be a smooth, strictly convex and reflexive Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $r > 0$  and let  $A \subset E \times E^*$  be a monotone operator such that  $D(A) \subset C \subset J^{-1}R(J + rA)$ . Then the resolvent  $Q_r$  of  $A$  defined by  $Q_rx = (J + rA)^{-1}Jx$  for all  $x \in C$  is a firmly nonexpansive type mapping, that is,*

$$(2.2) \quad \phi(Q_rx, Q_ry) + \phi(Q_ry, Q_rx) + \phi(Q_rx, x) + \phi(Q_ry, y) \leq \phi(Q_rx, y) + \phi(Q_ry, x)$$

for all  $x, y \in C$ .

**Corollary 2.5** ([17]). *Let  $C$  be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space  $E$ . Then the generalized projection  $\Pi_C$  from  $E$  onto  $C$  is a firmly nonexpansive type mapping, that is,*

$$(2.3) \quad \phi(\Pi_Cx, \Pi_Cy) + \phi(\Pi_Cy, \Pi_Cx) + \phi(\Pi_Cx, x) + \phi(\Pi_Cy, y) \leq \phi(\Pi_Cx, y) + \phi(\Pi_Cy, x)$$

for all  $x, y \in E$ .

### 3. THE EXISTENCE OF FIXED POINTS OF NONSPREADING MAPPINGS

In this section, we study the existence of fixed points of nonspreading mappings in Banach spaces. Using the technique developed by Takahashi [30], we can first show the following fixed point theorem for a single nonspreading mapping in Banach spaces:

**Theorem 3.1** ([16]). *Let  $E$  be a smooth, strictly convex and reflexive Banach space, let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a nonspreading mapping from  $C$  into itself. Then there exists  $x \in C$  such that  $\{T^n x\}$  is bounded if and only if  $T$  has a fixed point.*

As direct consequences of Theorem 3.1, we obtain the following corollaries:

**Corollary 3.2** ([16]). *Let  $E$  be a smooth, strictly convex and reflexive Banach space, let  $C$  be a nonempty bounded closed convex subset of  $E$  and let  $T$  be a nonspreading mapping from  $C$  into itself. Then  $T$  has a fixed point.*

**Corollary 3.3** ([16]). *Let  $H$  be a Hilbert space, let  $C$  be a nonempty closed convex subset of  $H$  and let  $T$  be a mapping from  $C$  into itself such that*

$$(3.1) \quad 2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all  $x, y \in C$ . Then there exists  $x \in C$  such that  $\{T^n x\}$  is bounded if and only if  $T$  has a fixed point.

By Corollary 2.2 and Theorem 3.1, we obtain the following:

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**Corollary 3.4** ([17]). *Let  $E$  be a smooth, strictly convex and reflexive Banach space, let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a firmly nonexpansive type mapping from  $C$  into itself. Then there exists  $x \in C$  such that  $\{T^n x\}$  is bounded if and only if  $T$  has a fixed point.*

We can also show the following common fixed point theorem for a commutative family of nonspreading mappings in Banach spaces:

**Theorem 3.5** ([16]). *Let  $E$  be a smooth, strictly convex and reflexive Banach space, let  $C$  be a nonempty bounded closed convex subset of  $E$  and let  $\{T_\alpha\}$  be a commutative family of nonspreading mappings from  $C$  into itself. Then  $\{T_\alpha\}$  has a common fixed point.*

## 4. THE ASYMPTOTIC BEHAVIOR OF FIRMLY NONEXPANSIVE TYPE MAPPINGS

In this section, we obtain a convergence theorem for a single firmly nonexpansive type mapping in Banach spaces (Theorem 4.4). To prove the result, we need the following crucial lemma:

**Lemma 4.1** ([16]). *Let  $E$  be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable, let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a nonspreading mapping from  $C$  into itself. Then  $\widehat{F}(T) = F(T)$ .*

Using Lemma 4.1, we can show the following theorems:

**Theorem 4.2** ([16]). *Let  $E$  be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable, let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a nonspreading mapping from  $C$  into itself such that  $F(T)$  is nonempty. Then  $T$  is a relatively nonexpansive mapping.*

**Theorem 4.3** ([17]). *Let  $E$  be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable, let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a firmly nonexpansive type mapping from  $C$  into itself such that  $F(T)$  is nonempty. Then  $T$  is a strongly relatively nonexpansive mapping.*

Using Theorem 4.3, we can prove the following convergence theorem:

**Theorem 4.4** ([17]). *Let  $E$  be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, let  $C$  be a nonempty closed convex subset of  $E$  and let  $T$  be a firmly nonexpansive type mapping from  $C$  into itself such that  $F(T)$  is nonempty. If  $J$  is weakly sequentially continuous, then for all  $x \in C$ , the sequence  $\{T^n x\}$  converges weakly to an element of  $F(T)$ .*

As a direct consequence of Theorem 4.4, we have the following result due to Martinet [19]:

**Corollary 4.5** ([19]). *Let  $C$  be a nonempty closed convex subset of a Hilbert space  $H$  and let  $T : C \rightarrow C$  be a firmly nonexpansive mapping such that  $F(T)$  is nonempty. Then for all  $x \in C$ , the sequence  $\{T^n x\}$  converges weakly to an element of  $F(T)$ .*

## 5. APPLICATIONS TO A PROXIMAL POINT ALGORITHM

In the final section, we apply our results to a proximal point algorithm for a monotone operator satisfying a range condition in Banach spaces. The *proximal point algorithm* was

originally proposed by Martinet [18] and generally studied by Rockafellar [28]. Let  $H$  be a Hilbert space and let  $A \subset H \times H$  be a maximal monotone operator. The proximal point algorithm generates a sequence  $\{x_n\}$  by  $x_1 = x \in H$  and  $x_{n+1} = J_{r_n}x_n$  for all  $n \in \mathbb{N}$ , where  $\{r_n\}$  is a sequence of positive real numbers and  $J_r$  is the resolvent of  $A$  defined by  $J_r = (I + rA)^{-1}$  for all  $r > 0$ .

By Corollaries 2.4, 3.4 and Theorem 4.4, we can show the following weak convergence theorem for a proximal point algorithm in Banach spaces; see [13, 14, 22] on similar results for *maximal* monotone operators in Banach spaces:

**Theorem 5.1** ([17]). *Let  $E$  be a smooth, strictly convex and reflexive Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $r > 0$  and let  $A \subset E \times E^*$  be a monotone operator such that  $D(A) \subset C \subset J^{-1}R(J + rA)$ . Let  $Q_r$  be the resolvent of  $A$  defined by  $Q_r z = (J + rA)^{-1}Jz$  for all  $z \in C$  and let  $\{x_n\}$  be a sequence defined by  $x_1 = x \in C$  and*

$$(5.1) \quad x_{n+1} = Q_r x_n$$

for all  $n \in \mathbb{N}$ . Then the following hold:

- (1) *The sequence  $\{x_n\}$  is bounded if and only if the set  $A^{-1}0$  is nonempty;*
- (2) *if  $A^{-1}0$  is nonempty,  $E$  is uniformly convex, the norm of  $E$  is uniformly Gâteaux differentiable and  $J$  is weakly sequentially continuous, then the sequence  $\{x_n\}$  converges weakly to an element of  $A^{-1}0$ .*

*Proof.* By Corollary 2.4,  $Q_r$  is a firmly nonexpansive type mapping from  $C$  into itself. We also know that  $F(Q_r) = A^{-1}0$ . Indeed, if  $u \in F(Q_r)$ , then we have  $Ju \in Ju + rAu$  and hence  $0 \in Au$ . On the other hand, if  $u \in A^{-1}0$ , then it follows from  $D(A) \subset C$  that  $u \in C$ . Since  $0 \in Au$ , we have  $Ju \in Ju + rAu$ . Hence we obtain  $Q_r u = u$ . Thus, by Corollary 3.4, if  $\{x_n\}$  is bounded, then  $A^{-1}0$  is nonempty; see [13, 14] for the converse implication. Thus the part (1) holds. By Theorem 4.4, the part (2) holds.  $\square$

In the particular case that the operator  $A$  is assumed to be maximal monotone, Theorem 5.1 is reduced to the following:

**Corollary 5.2.** *Let  $E$  be a smooth, strictly convex and reflexive Banach space and let  $A \subset E \times E^*$  be a maximal monotone operator. Let  $r > 0$ , let  $Q_r = (J + rA)^{-1}J$  and let  $\{x_n\}$  be a sequence defined by  $x_1 = x \in E$  and (5.1). Then the following hold:*

- (1) *The sequence  $\{x_n\}$  is bounded if and only if the set  $A^{-1}0$  is nonempty;*
- (2) *if  $A^{-1}0$  is nonempty,  $E$  is uniformly convex, the norm of  $E$  is uniformly Gâteaux differentiable and  $J$  is weakly sequentially continuous, then the sequence  $\{x_n\}$  converges weakly to an element of  $A^{-1}0$ .*

*Proof.* Since  $A$  is maximal monotone, by [3, 27], the equality  $R(J + rA) = E^*$  holds; see also [2, 31]. Thus the resolvent  $Q_r$  is a mapping from  $E$  into itself. By Theorem 5.1, we have the desired result.  $\square$

Let  $E$  be a Banach space and let  $f : E \rightarrow (-\infty, \infty]$  be a function. Then  $f$  is said to be *proper* if  $\{x \in E : f(x) \in \mathbb{R}\}$  is nonempty. The function  $f$  is said to be *lower semicontinuous* if  $\{x \in E : f(x) \leq r\}$  is closed in  $E$  for all  $r \in \mathbb{R}$ . The function  $f$  is also said to be *convex* if  $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$  whenever  $x, y \in E$  and

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$t \in (0, 1)$ . For a proper lower semicontinuous convex function, the *subdifferential*  $\partial f$  of  $f$  is defined by

$$(5.2) \quad \partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y), \forall y \in E\}$$

for all  $x \in E$ . It is known that if  $f : E \rightarrow (-\infty, \infty]$  is proper, lower semicontinuous and convex and  $g : E \rightarrow \mathbb{R}$  is continuous and convex, then

$$(5.3) \quad \partial(f + g) = \partial f + \partial g.$$

We denote the set of minimizers of  $f : E \rightarrow (-\infty, \infty]$  by  $\arg \min_{y \in E} f(y)$ .

Using Corollary 5.2, we can study the problem of finding minimizers of proper lower semicontinuous convex functions in Banach spaces:

**Corollary 5.3.** *Let  $E$  be a smooth, strictly convex and reflexive Banach space and let  $f : E \rightarrow (-\infty, \infty]$  be a proper lower semicontinuous convex function. Let  $r > 0$  and let  $\{x_n\}$  be a sequence defined by  $x_1 = x \in E$  and*

$$(5.4) \quad x_{n+1} = \arg \min_{y \in E} \left\{ f(y) + \frac{1}{2r} \phi(y, x_n) \right\}$$

for all  $n \in \mathbb{N}$ . Then the following hold:

- (1) *The sequence  $\{x_n\}$  is bounded if and only if the set  $\arg \min_{y \in E} f(y)$  is nonempty;*
- (2) *if  $\arg \min_{y \in E} f(y)$  is nonempty,  $E$  is uniformly convex, the norm of  $E$  is uniformly Gâteaux differentiable and  $J$  is weakly sequentially continuous, then the sequence  $\{x_n\}$  converges weakly to an element of  $\arg \min_{y \in E} f(y)$ .*

*Proof.* By Rockafellar's theorem [25, 26], the subdifferential mapping  $\partial f$  of  $f$  is maximal monotone. It is also known that  $(\partial f)^{-1}(0) = \arg \min_{y \in E} f(y)$ .

Let  $Q_r = (J + r\partial f)^{-1}J$ . For each  $x \in E$ , it follows from (5.3) that

$$(5.5) \quad z = Q_r x \iff 0 \in \partial \left( f + \frac{1}{2r} \phi(\cdot, x) \right) (z) \iff z = \arg \min_{y \in E} \left\{ f(y) + \frac{1}{2r} \phi(y, x) \right\}$$

Thus we obtain  $x_{n+1} = Q_r x_n$  for all  $n \in \mathbb{N}$ . Hence, by Corollary 5.2, we have the desired result.  $\square$

## REFERENCES

- [1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and applications of nonlinear operators of accretive and monotone type (A. G. Kartsatos Ed.), Lecture Notes in Pure and Appl. Math. **178**, Dekker, New York, 1996, pp.15–50.
- [2] V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Editura Academiei Republicii Socialiste România, Bucharest (1976).
- [3] F. E. Browder, *Nonlinear maximal monotone operators in Banach space*, Math. Ann. **175** (1968), 89–113.
- [4] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math. **47** (1973), 341–355.
- [5] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459–470.
- [6] D. Butnariu and A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers, Dordrecht (2000).
- [7] D. Butnariu, S. Reich and A. J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces*, J. Appl. Anal. **7** (2001), 151–174.
- [8] D. Butnariu, S. Reich and A. J. Zaslavski, *Weak convergence of orbits of nonlinear operators in reflexive Banach spaces*, Numer. Funct. Anal. Optim. **24** (2003), 489–508.



- [9] Y. Censor and S. Reich, *Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization*, Optimization **37** (1996), 323–339.
- [10] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers Group, Dordrecht (1990).
- [11] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge (1990).
- [12] W. Kaczor, *Fixed points of  $\lambda$ -firmly nonexpansive mappings on nonconvex sets*, Nonlinear Anal. **47** (2001), 2787–2792.
- [13] S. Kamimura, *The proximal point algorithm in a Banach space*, Proceedings of the Third International Conference on Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2004, pp.143–148.
- [14] S. Kamimura, F. Kohsaka and W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*, Set-Valued Anal. **12** (2004), 417–429.
- [15] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.
- [16] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, submitted.
- [17] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive type mappings in Banach spaces*, submitted.
- [18] B. Martinet, *Régularisation d'inéquations variationnelles par approximations successives*, Rev. Française Informat. Recherche Opérationnelle **4** (1970), 154–158.
- [19] B. Martinet, *Détermination approchée d'un point fixe d'une application pseudo-contractante. Cas de l'application prox*, C. R. Acad. Sci. Paris Sér. A-B **274** (1972), A163–A165.
- [20] S. Matsushita and W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2004** (2004), 37–47.
- [21] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory **134** (2005), 257–266.
- [22] S. Matsushita and W. Takahashi, *The existence of zeros of monotone operators concerning optimization problems*, Sūrikaiseikikenkyūsho Kōkyūroku, Kyoto **1461** (2005), 40–46 (Japanese).
- [23] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, Theory and applications of nonlinear operators of accretive and monotone type (A. G. Kartsatos Ed.), Lecture Notes in Pure and Appl. Math. **178**, Dekker, New York, 1996, pp.313–318.
- [24] S. Reich and I. Shafrir, *The asymptotic behavior of firmly nonexpansive mappings*, Proc. Amer. Math. Soc. **101** (1987), 246–250.
- [25] R. T. Rockafellar, *Characterization of the subdifferentials of convex functions*, Pacific J. Math. **17** (1966), 497–510.
- [26] R. T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. **33** (1970), 209–216.
- [27] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.
- [28] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.
- [29] R. Smarzewski, *On firmly nonexpansive mappings*, Proc. Amer. Math. Soc. **113** (1991), 723–725.
- [30] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.
- [31] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama (2000) (Japanese).
- [32] W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama (2000).

## FIXED POINT THEOREMS FOR NONLINEAR MAPPINGS

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